Chp 4

Big Ideas

- Boundary conditions lead to quantization

- Quantum numbers are added w/ each dimension

- Observables corresponding to operators for which the wavefunction is not an eigenfunction are in a superposition of eigenstates until measured.

- Nodes increase w/ quantum #
Skills

- Solve the 1-D free particle
- Solve the 1-D PIB
- Normalize the 1-D - PIB
Model System: The 1-D Free Particle

Free: \( V = 0 \)

\[ 1-D \quad \hat{A} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \hat{x} \]

\( \hat{A} \psi = E \psi \)

\[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = E \cdot \psi \]

or

\[ \frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2} \psi \]

\[ \frac{\partial^3 \psi}{\partial x^3} + 0 \cdot \frac{\partial \psi}{\partial x^2} + \frac{2mE}{\hbar^2} \psi = 0 \]

This type of differential equation is easy to solve.
\[ Ay'' + By' + Cy = 0 \]

1) characteristic eq
   \[ Am^2 + Bm + C = 0 \]
   solve for the roots: \( r_1, r_2 \)

2) \[ y = C_1 e^{r_1 x} + C_2 e^{r_2 x} \]

3) \( C_1 \) and \( C_2 \) are constants determined by boundary conditions
for 1-D free particle < lets use $s$ for characteristic eq >

\[ S^2 + \frac{2mE}{\hbar^2} = 0 \]

\[ S^2 = \frac{2mE}{\hbar^2} \Rightarrow S = \pm \sqrt{\frac{2mE}{\hbar^2}} \]

\[ S = \pm \left( \frac{1}{\hbar} \sqrt{2mE} \right) \quad \text{so} \]

\[ \psi = C_1 e^{\frac{i}{\hbar} \sqrt{2mE} x} + C_2 e^{-\frac{i}{\hbar} \sqrt{2mE} x} \]

we could write this as

\[ \psi = C_1 \psi_+ + C_2 \psi_- \]
Note $\Psi$ is not an eigenfunction of $P_x$, but $\Psi_-$ and $\Psi_+$ are. For instance:

\[ P_x \Psi_+ = 0 \quad \text{and} \quad -i \hbar \frac{2}{2x} \Psi_+ = -i \hbar \left( e^{\frac{i}{\hbar} \sqrt{2mE} x} \right) \left( \frac{i}{\hbar} \sqrt{2mE} \Psi_+ \right) \]

so

\[ P_x = (-i \hbar)(i \hbar \sqrt{2mE}) \]

\[ P_x = \sqrt{2mE} \quad (\text{or} \quad mv) \]

for $\Psi_-$ it would be $-mv$.

The general solution, $\Psi$ represents a superposition of states.

Note also that $E$ is not quantized.
The 1-D PIB

\[ \begin{align*}
0 & \leq x \leq 1 \\
V & = 0 \\
\text{if } x < 0 \text{ or } x > 1 \\
V & = \infty
\end{align*} \]

\[ A = -\frac{\hbar^2 \frac{\partial^2}{\partial x^2}}{2m} \quad \text{(inside box, } V = 0) \]

\[ i \hbar \sqrt{2\pi E} \quad -i \hbar \sqrt{2\pi E} \]

So \[ \psi = c_1 e^{-} + c_2 e^{+} \]

\[ A = i (c_1 - c_2) \quad \text{ define } k = \frac{\sqrt{2\pi E}}{\hbar} \]

\[ \psi = A \sin(kx) + B \cos(kx) \]

Note \[ \psi(0) = 0, \]

So \[ B = 0 \]

\[ \psi = A \sin(kx) \]